

ENDOMORPHISM ALGEBRAS AND Q-TRACES

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ABSTRACT. For a braided vector space (V, σ) with braiding σ of Hecke type, we introduce three associative algebra structures on the space $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$ of graded endomorphisms of the quantum symmetric algebra $S_\sigma(V)$. We use the second product to construct a new trace. This trace is an algebra morphism with respect to the third product. In particular, when V is the fundamental representation of $\mathcal{U}_q \mathfrak{sl}_{N+1}$ and σ is the action of the R -matrix, this trace is a scalar multiple of the quantum trace of type A .

1. INTRODUCTION

More than twenty years ago, H. Osborn studied the space $\oplus_{i \geq 0} \text{End} \bigwedge^i(V)$ of graded endomorphisms of the exterior algebra in order to give an algebraic construction of Chern-Weil theory ([8, 9]). He introduced three associative products on this space. The first one is just the composition of endomorphisms. Since the exterior algebra is also a coalgebra, he defined the second one to be the convolution product. And then he combined the first two ones to construct the third product. Assuming that $\dim \bigwedge^i(V) = 1$ for sufficiently large i , he constructed a trace function by using the second product. This trace gives the usual one when it is restricted on $\text{End}(V)$. And it is an algebra morphism when one considers the third product.

On the other side, after the creation of quantum groups by Drinfel'd [1] and Jimbo [5], mathematicians use Yang-Baxter operators to quantize various classical objects in algebra and find many interesting phenomena. Since symmetric algebras and exterior algebras are defined by using flips which are trivial Yang-Baxter operators, it seems quite reasonable and possible to quantize them. In his paper [3], Gurevich studied Yang-Baxter operators of Hecke type, which he called Hecke symmetries. And then he defined the symmetric algebra and the exterior algebra with respect to these operators. They are analogue to the usual ones. Later, different aspects of these algebras were discussed in [4] and [12]. In [11], a very remarkable property of the quantized symmetric algebra was discovered. For some special Yang-Baxter operators, the symmetric one, as Hopf algebra, is isomorphic to the "upper triangular part" of the quantized enveloping algebra associated with a symmetrizable Cartan matrix.

Naturally, it is interesting to see what will happen when one extends Osborn's trace to the quantum case. Let (V, σ) be a braided vector space with braiding σ of Hecke type, and $S_\sigma^p(V)$ be the p -th component of the quantum symmetric algebra $S_\sigma(V)$ built on (V, σ) . We assume that $\dim S_\sigma^M(V) = 1$ for some M and

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$\dim S_\sigma^p(V) = 0$ for $p > M$. Then, on the vector space $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$, the convolution product, the third product and the trace can be constructed step-by-step following the ones in [9]. And this trace, called q-trace, is an algebra morphism with respect to the third product. In particular, let V be the fundamental representation of $\mathcal{U}_q \mathfrak{sl}_{N+1}$ and σ the braiding given by the R -matrix of $\mathcal{U}_q \mathfrak{sl}_{N+1}$. Then σ is of Hecke type and $S_\sigma^p(V)$ vanishes when p is sufficiently large. To our surprise, the q-trace in this case has already existed for more than one decade.

In the theory of quantum groups, there is an important invariant which generalizes the usual trace of endomorphisms. It is the so-called quantum trace. Let \mathcal{C} be a ribbon category with unit I , V be an object of \mathcal{C} and f be an endomorphism of V . The quantum trace $\text{tr}_q(f)$ of f is an element in the monoid $\text{End}(I)$ (see, e.g., [6]). It coincides with the usual trace when $\mathcal{C} = \text{Vect}(k)$. When we take \mathcal{C} to be the category of finite dimensional representations of u_ϵ (for the definition, one can see [7]), the quantum trace is given by composing the usual trace with the action of the group-like elements K_i 's. This is a functorial approach. After an easy computation, we can show that our q-trace is a scalar multiple of the quantum trace. So we get a more elementary approach to the quantum trace of type A .

This paper is organized as follows. In Section 2 we define the three products on $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$ for a braided vector space (V, σ) with a braiding σ of Hecke type. Then we construct the q-trace of $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$ and prove that it is an algebra morphism with respect to the third product. In Section 3, we apply our constructions to the special braided vector space (V, σ) , where V is the fundamental representation of $\mathcal{U}_q \mathfrak{sl}_{N+1}$ and σ is the braiding given by the R -matrix of $\mathcal{U}_q \mathfrak{sl}_{N+1}$. We prove that the q-trace for this case is just a scalar multiple of the quantum trace of $\mathcal{U}_q \mathfrak{sl}_{N+1}$.

NOTATION

We fix our ground field to be the complex number field \mathbb{C} .

We denote by \mathfrak{S}_p the symmetric group of $\{1, \dots, p\}$. For $\{i_1, \dots, i_k\} \subset \{1, \dots, p\}$, we denote $l(i_1, \dots, i_k) = \#\{(i_s, i_t) | 1 \leq s < t \leq k, i_s > i_t\}$. And for any $w \in \mathfrak{S}_p$, $l(w) = l(w(1), \dots, w(p))$. It is just the length of w .

An (i, j) -shuffle is an element $w \in \mathfrak{S}_{i+j}$ such that $w(1) < \dots < w(i)$ and $w(i+1) < \dots < w(i+j)$. We denote by $\mathfrak{S}_{i,j}$ the set of all (i, j) -shuffles.

Let V be a vector space. A braiding σ on V is an invertible linear map in $\text{End}(V \otimes V)$ satisfying the *quantum Yang-Baxter equation*:

$$(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

A braided vector space (V, σ) is a vector space V equipped with a braiding σ . For any $p \in \mathbb{N}$ and $1 \leq i \leq p-1$, we denote by σ_i the operator $\text{id}^{\otimes(i-1)} \otimes \sigma \otimes \text{id}^{\otimes(p-i-1)} \in \text{End}(V^{\otimes p})$. For any $w \in \mathfrak{S}_p$, we denote by T_w the corresponding lift of w in the braid group B_p , defined as follows: if $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression of w , where $s_i = (i, i+1)$, then $T_w = \sigma_{i_1} \cdots \sigma_{i_l}$. We also use T_w^σ to indicate the action of σ .

Let q be a nonzero number in \mathbb{C} . For $q \neq 1$ and any $n = 0, 1, 2, \dots$, we denote $(n)_q = (1 - q^n)/(1 - q)$, and

$$(n)_q! = \begin{cases} 1, & n = 0, \\ \frac{(1-q) \cdots (1-q^n)}{(1-q)^n}, & n \geq 1. \end{cases}$$

2. THE Q-TRACE

In this section, we define the q-trace and prove that it is an algebra morphism with respect to the third product. We start by recalling some notions and properties of braidings of Hecke type and quantum symmetric algebras for the later use. For more details, one can see [2], [3], [11] and [12].

2.1. Braidings of Hecke type and quantum symmetric algebras. Let (V, σ) be a braided vector space. The braiding σ is said to be of *Hecke type* if it satisfies the following *Iwahori's quadratic equation*:

$$(\sigma + \text{id}_{V \otimes V})(\sigma - \nu \text{id}_{V \otimes V}) = 0,$$

where ν is a nonzero scalar in \mathbb{C} .

In the rest of this section, σ is always a braiding of Hecke type with parameter $\nu \in \mathbb{C}^*$.

For $p \geq 1$, we define $A^{(p)} = \sum_{w \in \mathfrak{S}_p} T_w^\sigma$. The following proposition of $A^{(p)}$ plays an essential role in the construction of q-trace. It is due to D. I. Gurevich ([3], Proposition 2.4).

Proposition 2.1. *For $p \geq 1$ we have*

$$(A^{(p)})^2 = (p)_\nu! A^{(p)}.$$

The image of the map $\oplus_{p \geq 0} A^{(p)}$ has important algebraic structures on it. The first one is the quantum shuffle product which was introduced by M. Rosso [10, 11]. It generalizes the usual shuffle product on $T(V)$. For any $v_1, \dots, v_{i+j} \in V$, the quantum shuffle product sh is defined to be

$$sh((v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_{i+j})) = \sum_{w \in \mathfrak{S}_{i,j}} T_w(v_1 \otimes \cdots \otimes v_{i+j}).$$

We denote by $T_\sigma(V)$ the quantum shuffle algebra $(T(V), sh)$. The subalgebra $S_\sigma(V)$ of $T_\sigma(V)$ generated by V with respect to the quantum shuffle product is called the *quantum symmetric algebra*. It is easy to see that $S_\sigma(V) = \oplus_{p \geq 0} \text{Im}(\sum_{w \in \mathfrak{S}_p} T_w)$. We denote by $S_\sigma^p(V) = \text{Im}(\sum_{w \in \mathfrak{S}_p} T_w)$ the p -th component of $S_\sigma(V)$.

The algebra $T_\sigma(V)$ is a coalgebra with the deconcatenation coproduct δ :

$$\delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^n (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

We denote by δ_{ij} the composition of δ with the projection $T(V) \otimes T(V) \rightarrow V^{\otimes i} \otimes V^{\otimes j}$. One can show that $(S_\sigma(V), \delta)$ is also a coalgebra ([11]).

2.2. Algebraic structures on $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$. Let σ be a braiding of Hecke type on V such that $\dim S_\sigma^M(V) = 1$ for some M and $\dim S_\sigma^p(V) = 0$ for $p > M$. For $\mathbf{A} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$, we write $\mathbf{A} = (A_0, A_1, \dots, A_M)$, where $A_p \in \text{End} S_\sigma^p(V)$ is the p -th component of \mathbf{A} .

For $\mathbf{A}, \mathbf{B} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$, we define the *composition product* $\mathbf{A} \circ \mathbf{B} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$ by $(\mathbf{A} \circ \mathbf{B})_p = A_p \circ B_p$ with the usual composition. Obviously, $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$ is an associative algebra with the two-sided unit element $\mathbf{I} = (\mathbf{I}_0, \mathbf{I}_1, \dots, \mathbf{I}_M)$, where \mathbf{I}_p is the identity map of $S_\sigma^p(V)$.

We can also define the *convolution product* $\mathbf{A} * \mathbf{B} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$ by

$$(\mathbf{A} * \mathbf{B})_p = \sum_{l=0}^p A_l * B_{p-l},$$

where $A_i * B_j = sh \circ (A_i \otimes B_j) \circ \delta_{i,j} \in \text{End} S_\sigma^{i+j}(V)$. It is well-known that the convolution product of endomorphisms is associative. It follows immediately that $(\oplus_{p=0}^M \text{End} S_\sigma^p(V), *)$ is an associative algebra with the two-sided unit element $\mathbf{I}_0 = (\mathbf{I}_0, 0, \dots, 0)$.

Proposition 2.2. *For $0 \leq p \leq M$, we have*

$$\mathbf{I}_1^{*p} = (p)_\nu! \mathbf{I}_p.$$

Proof. We first notice that for any $v_1, \dots, v_p \in V$,

$$sh(v_1 \otimes sh(v_2 \otimes \dots \otimes sh(v_{p-1} \otimes v_p) \dots)) = A^{(p)}(v_1 \otimes \dots \otimes v_p).$$

Then

$$\begin{aligned} \mathbf{I}_1^{*p} \circ A^{(p)} &= A^{(p)} \circ \mathbf{I}_1^{\otimes p} \circ A^{(p)} \\ &= (A^{(p)})^2 \\ &= (p)_\nu! A^{(p)}. \end{aligned}$$

□

Corollary 2.3. *For $0 \leq i, j \leq M$ with $i + j \leq M$, we have*

$$\mathbf{I}_i * \mathbf{I}_j = \binom{i+j}{i}_\nu \mathbf{I}_{i+j},$$

where $\binom{i+j}{i}_\nu = (i+j)_\nu! / ((i)_\nu! (j)_\nu!)$.

Now we assume that the parameter ν in the Iwahori's equation is not a root of unity. For any $A \in \text{End} S_\sigma^1(V) = \text{End}(V)$, we define

$$e_\nu^{*A} = (\mathbf{I}_0, \frac{1}{(1)_\nu!} A, \frac{1}{(2)_\nu!} A^{*2}, \dots, \frac{1}{(M)_\nu!} A^{*M}).$$

In particular, $e_\nu^{*\mathbf{I}_1} = (\mathbf{I}_0, \mathbf{I}_1, \dots, \mathbf{I}_M)$.

If we write

$$(e_\nu^{*A})^{-1} = (\mathbf{I}_0, \frac{-1}{(1)_\nu!} A, \frac{\nu}{(2)_\nu!} A^{*2}, \dots, \frac{(-1)^M \nu^{M(M-1)/2}}{(M)_\nu!} A^{*M}),$$

then

$$(e_\nu^{*A})^{-1} * e_\nu^{*A} = e_\nu^{*A} * (e_\nu^{*A})^{-1} = \mathbf{I}_0.$$

We define

$$\begin{aligned} \alpha : \oplus_{p=0}^M \text{End} S_\sigma^p(V) &\rightarrow \oplus_{p=0}^M \text{End} S_\sigma^p(V), \\ \mathbf{A} &\mapsto \mathbf{A} * e_\nu^{*I_1}. \end{aligned}$$

Consequently, α has an inverse defined by $\alpha^{-1}(\mathbf{A}) = \mathbf{A} * (e_\nu^{*I_1})^{-1}$.

Definition 2.4. For any $\mathbf{A}, \mathbf{B} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$, the third product $\mathbf{A} \times \mathbf{B}$ of \mathbf{A} and \mathbf{B} is defined to be

$$\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha \mathbf{A}) \circ (\alpha \mathbf{B})) = ((\mathbf{A} * e_\nu^{*I_1}) \circ (\mathbf{B} * e_\nu^{*I_1})) * (e_\nu^{*I_1})^{-1}.$$

Proposition 2.5. The space $\oplus_{p=0}^M \text{End} S_\sigma^p(V)$ equipped with the third product is an associative algebra with two-sided unit element I_0 .

Proof. For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$, we have

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= \alpha^{-1}((\alpha(\mathbf{A} \times \mathbf{B})) \circ (\alpha \mathbf{C})) \\ &= \alpha^{-1}((\alpha \circ \alpha^{-1}((\alpha \mathbf{A}) \circ (\alpha \mathbf{B}))) \circ (\alpha \mathbf{C})) \\ &= \alpha^{-1}((\alpha \mathbf{A}) \circ (\alpha \mathbf{B}) \circ (\alpha \mathbf{C})) \\ &= \mathbf{A} \times (\mathbf{B} \times \mathbf{C}). \end{aligned}$$

And

$$\begin{aligned} I_0 \times \mathbf{A} &= \alpha^{-1}((\alpha I_0) \circ (\alpha \mathbf{A})) \\ &= \alpha^{-1}((I_0 * e_\nu^{*I_1}) \circ (\alpha \mathbf{A})) \\ &= \alpha^{-1}(e_\nu^{*I_1} \circ (\alpha \mathbf{A})) \\ &= \alpha^{-1}(\alpha \mathbf{A}) \\ &= \mathbf{A}. \end{aligned}$$

Similarly, we have that $\mathbf{A} \times I_0 = \mathbf{A}$. \square

Proposition 2.6. For $0 \leq r \leq M$, $A_i \in \text{End} S_\sigma^i(V)$ and $B_j \in \text{End} S_\sigma^j(V)$, we have

$$(A_i \times B_j)_r = \sum_{s=0}^r \frac{\nu^{s(s-1)/2}}{(s)_\nu!} ((A_i * I_{r-s-i}) \circ (B_j * I_{r-s-j})) * I_1^{*s},$$

where $I_t = 0$ for $t < 0$.

Proof. The formula follows from the definition of the third product. \square

Corollary 2.7. We have $(A_i \times B_j)_r = 0$ for $r < \max(i, j)$ and $(A_r \times B_r)_r = A_r \circ B_r$.

2.3. The q-trace.

Definition 2.8. The q-trace of any $\mathbf{A} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$ is the unique element $\text{Tr}_q \mathbf{A} \in \mathbb{C}$ such that $(\alpha \mathbf{A})_M = (\text{Tr}_q \mathbf{A}) I_M \in \text{End} S_\sigma^M(V)$.

Theorem 2.9. The q-trace is an algebra morphism with respect to the third product. Precisely, for $\mathbf{A}, \mathbf{B} \in \oplus_{p=0}^M \text{End} S_\sigma^p(V)$, we have

$$1. \text{Tr}_q(\mathbf{A} + \mathbf{B}) = \text{Tr}_q \mathbf{A} + \text{Tr}_q \mathbf{B},$$

2. $\text{Tr}_q(\mathbf{A} \times \mathbf{B}) = (\text{Tr}_q \mathbf{A})(\text{Tr}_q \mathbf{B})$,
3. $\text{Tr}_q(\mathbf{A} \times \mathbf{B}) = \text{Tr}_q(\mathbf{B} \times \mathbf{A})$.

Proof. 1. By the definition, we have

$$\begin{aligned} (\alpha(\mathbf{A} + \mathbf{B}))_M &= \sum_{k=0}^M (A_k + B_k) * \mathbf{I}_{M-k} \\ &= (\alpha \mathbf{A})_M + (\alpha \mathbf{B})_M. \end{aligned}$$

So

$$\begin{aligned} \text{Tr}_q(\mathbf{A} + \mathbf{B})\mathbf{I}_M &= (\text{Tr}_q \mathbf{A})\mathbf{I}_M + (\text{Tr}_q \mathbf{B})\mathbf{I}_M \\ &= (\text{Tr}_q \mathbf{A} + \text{Tr}_q \mathbf{B})\mathbf{I}_M. \end{aligned}$$

Therefore $\text{Tr}_q(\mathbf{A} + \mathbf{B}) = \text{Tr}_q \mathbf{A} + \text{Tr}_q \mathbf{B}$.

2. Since $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha \mathbf{A}) \circ (\alpha \mathbf{B}))$, we have $\alpha(\mathbf{A} \times \mathbf{B}) = (\alpha \mathbf{A}) \circ (\alpha \mathbf{B})$. So

$$(\alpha(\mathbf{A} \times \mathbf{B}))_M = (\alpha \mathbf{A})_M \circ (\alpha \mathbf{B})_M,$$

which implies that

$$\begin{aligned} \text{Tr}_q(\mathbf{A} \times \mathbf{B})\mathbf{I}_M &= (\text{Tr}_q \mathbf{A})\mathbf{I}_M \circ (\text{Tr}_q \mathbf{B})\mathbf{I}_M \\ &= (\text{Tr}_q \mathbf{A})(\text{Tr}_q \mathbf{B})\mathbf{I}_M. \end{aligned}$$

So we have $\text{Tr}_q(\mathbf{A} \times \mathbf{B}) = (\text{Tr}_q \mathbf{A})(\text{Tr}_q \mathbf{B})$.

3. It follows from the identity stated in 2 immediately. \square

3. ANOTHER APPROACH OF THE QUANTUM TRACE

In the previous section we have defined the third product and the q-trace in a general setting. In this section, we study a special case of braided vector spaces which provides an elementary approach to the quantum trace of type A . We first introduce the quantum exterior algebra which is the quantum symmetric algebra related to the fundamental representation of $\mathcal{U}_q \mathfrak{sl}_{N+1}$. And then we give a more explicit law for the second product in this case. Using the computational result, we give the formula of the q-trace and compare it with the quantum trace of type A .

3.1. Quantum exterior algebras. In the rest of this paper, we denote $V = \mathbb{C}^{N+1}$ and by E_{ij} the matrix with entry 1 in the position (i, j) and entries 0 elsewhere. The *fundamental representation* of $\mathcal{U}_q \mathfrak{sl}_{N+1}$ is the algebra homomorphism

$$\begin{aligned} \rho : \mathcal{U}_q \mathfrak{sl}_{N+1} &\rightarrow \text{End} V, \\ E_i &\mapsto E_{i, i+1}, \\ F_i &\mapsto E_{i+1, i}, \\ K_i &\mapsto \sum_{l \neq i, i+1} E_{il} + qE_{ii} + q^{-1}E_{i+1, i+1}, \end{aligned}$$

where E_i 's, F_i 's and K_i 's are the standard generators of $\mathcal{U}_q \mathfrak{sl}_{N+1}$.

Then the action of the R -matrix on $V \otimes V$ is given by

$$R_\rho = q \sum_{i=1}^{N+1} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{i < j} E_{jj} \otimes E_{ii}.$$

Let $c = q^{-1}R_\rho \in \text{GL}(V \otimes V)$. If we denote by $e_i = (0, \dots, 0, 1, 0, \dots, 0)^t \in V$ the unit column vector whose components are zero except the i -th component is 1, then we have:

$$c(e_i \otimes e_j) = \begin{cases} e_i \otimes e_i, & i = j, \\ q^{-1}e_j \otimes e_i, & i < j, \\ q^{-1}e_j \otimes e_i + (1 - q^{-2})e_i \otimes e_j, & i > j. \end{cases}$$

The map c is a braiding on V and satisfies the *Iwahori's quadratic equation*:

$$(c - \text{id}_{V \otimes V})(c + q^{-2}\text{id}_{V \otimes V}) = 0.$$

Definition 3.1. Let \mathfrak{I} be the two-sided ideal generated by $\text{Ker}(\text{id}_{V \otimes V} - c)$ in $T(V)$. The quotient algebra $\bigwedge_c(V) = T(V)/\mathfrak{I}$ is called the quantum exterior algebra on V .

By an easy computation, we have

$$\text{Ker}(\text{id}_{V \otimes V} - c) = \text{Span}_{\mathbb{C}}\{e_i \otimes e_i, q^{-1}e_i \otimes e_j + e_j \otimes e_i (i < j)\}.$$

Let $\pi : T(V) \rightarrow \bigwedge_c(V)$ be the canonical projection. For any $e_{i_1} \otimes \dots \otimes e_{i_p} \in T^p(V)$, we denote $e_{i_1} \wedge \dots \wedge e_{i_p} = \pi(e_{i_1} \otimes \dots \otimes e_{i_p})$. It follows immediately that:

1. The algebra $\bigwedge_c(V)$ is graded and generated by $\{e_1, \dots, e_{N+1}\}$ with the relations:

$$e_i \wedge e_i = 0,$$

and

$$e_j \wedge e_i = -q^{-1}e_i \wedge e_j \ (i < j).$$

2. If we denote by $\bigwedge_c^p(V)$ the p -th component of $\bigwedge_c(V)$, then $\dim \bigwedge_c^{N+1}(V) = 1$ and $\bigwedge_c^p(V) = 0$ for $p > N + 1$.

3. The set $\{e_{i_1} \wedge \dots \wedge e_{i_p} | 1 \leq i_1 < \dots < i_p \leq N + 1, 1 \leq p \leq N + 1\}$ forms a linear basis of $\bigwedge_c(V)$.

Let $A^{(p)} = \sum_{w \in \mathfrak{S}_p} T_w^{-c}$. Then by the following proposition, we can view the quantum exterior algebra as a special quantum symmetric algebra.

Proposition 3.2 ([3], Proposition 2.13). *For $k \geq 1$, we have the following linear isomorphism:*

$$\text{Im} A^{(k)} \cong \bigwedge_c^k(V).$$

We can identify $\bigwedge_c(V)$ with $S_{-c}(V)$ as linear space. Moreover, since $sh(A^{(i)} \otimes A^{(j)}) = \sum_{w \in \mathfrak{S}_{i,j}} T_w^{-c}(A^{(i)} \otimes A^{(j)}) = A^{i+j}$, the product in $\bigwedge_c(V)$ is just the quantum shuffle product in $S_{-c}(V)$. The space $\bigwedge_c(V)$ also inherits the coproduct of $S_{-c}(V)$. It is not difficult to show the following formula for the deconcatenation coproduct on $\bigwedge_c(V)$: for $1 \leq t \leq p \leq N + 1$ and $1 \leq i_1 < i_2 < \dots < i_p \leq N + 1$,

$$\delta_{t,p-t}(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{w \in \mathfrak{S}_{t,p-t}} (-q)^{-l(w)} e_{i_{w(1)}} \wedge \dots \wedge e_{i_{w(t)}} \otimes e_{i_{w(t+1)}} \wedge \dots \wedge e_{i_{w(p)}}.$$

3.2. Explicit law of the second product. In order to give an explicit formula of the q-trace, we describe the convolution product more precisely in our special case.

We define $c^\vee = (c^{-1})^t$, where t means the transpose of the operator. Then $c^\vee \in \text{GL}(V^* \otimes V^*)$. Let $\{f_i\}$ be the dual basis of $\{e_i\}$. We have

$$c^\vee(f_i \otimes f_j) = \begin{cases} f_i \otimes f_i, & i = j, \\ qf_j \otimes f_i + (1 - q^2)f_i \otimes f_j, & i < j, \\ qf_j \otimes f_i, & i > j. \end{cases}$$

Obviously c^\vee is a braiding on V^* and satisfies the Iwahori's equation:

$$(c^\vee - \text{id}_{V^* \otimes V^*})(c^\vee + q^2 \text{id}_{V^* \otimes V^*}) = 0.$$

It is easy to show that $\bigwedge_{c^\vee}(V^*)$, as an algebra, is generated by f_i 's with the relations:

$$f_i \wedge f_i = 0, \quad f_j \wedge f_i = -q^{-1}f_i \wedge f_j \quad (i < j).$$

Therefore the map $e_i \mapsto f_i$ induces an isomorphism of algebras: $\bigwedge_c(V) \rightarrow \bigwedge_{c^\vee}(V^*)$.

For any $s < t$, we have

$$\begin{aligned} E_{ij} * E_{kl}(e_s \wedge e_t) &= sh \circ (E_{ij} \otimes E_{kl}) \circ \delta_{1,1}(e_s \wedge e_t) \\ &= sh \circ (E_{ij} \otimes E_{kl})(e_s \otimes e_t - q^{-1}e_t \otimes e_s) \\ &= \delta_{js}\delta_{lt}e_i \wedge e_k - q^{-1}\delta_{jt}\delta_{ls}e_i \wedge e_k \\ &= (\delta_{js}\delta_{lt} - q^{-1}\delta_{jt}\delta_{ls})e_i \wedge e_k. \end{aligned}$$

Similarly, $E_{kl} * E_{ij}(e_s \wedge e_t) = (\delta_{ls}\delta_{jt} - q^{-1}\delta_{lt}\delta_{js})e_k \wedge e_i$. So we get that

$$\begin{cases} E_{ij} * E_{ik} &= E_{ij} * E_{kj} = 0, & \forall i, j, k, \\ E_{kj} * E_{il} &= -q^{-1}E_{ij} * E_{kl}, & \text{if } i < k, \forall j, l, \\ E_{il} * E_{kj} &= -q^{-1}E_{ij} * E_{kl}, & \text{if } j < l, \forall i, k. \end{cases} \quad (1)$$

In general, for $1 \leq i_1 < \dots < i_p \leq N+1$, $1 \leq j_1 < \dots < j_p \leq N+1$ and $1 \leq l_1 < \dots < l_p \leq N+1$, we have

$$E_{i_1 j_1} * \dots * E_{i_p j_p}(e_{l_1} \wedge \dots \wedge e_{l_p}) = \begin{cases} e_{i_1} \wedge \dots \wedge e_{i_p}, & \text{if } j_k = l_k, \\ 0, & \text{otherwise.} \end{cases}$$

So the set

$$\{E_{i_1 j_1} * \dots * E_{i_p j_p} \mid 1 \leq i_1 < \dots < i_p \leq N+1, 1 \leq j_1 < \dots < j_p \leq N+1\}$$

forms a linear basis of $\text{End } \bigwedge_c^p(V)$, which implies that $\bigoplus_{p=0}^{N+1} \text{End } \bigwedge_c^p(V)$ is an algebra generated by $\{E_{ij}\}$. As a consequence, we have the following linear isomorphism:

$$\begin{aligned} \iota_p : \quad \text{End } \bigwedge_c^p(V) &\rightarrow \bigwedge_c^p(V) \otimes \bigwedge_{c^\vee}^p(V^*), \\ E_{i_1 j_1} * \dots * E_{i_p j_p} &\mapsto e_{i_1} \wedge \dots \wedge e_{i_p} \otimes f_{j_1} \wedge \dots \wedge f_{j_p}. \end{aligned}$$

If we endow $\bigwedge_c(V) \otimes \bigwedge_{c^\vee}(V^*)$ with the tensor algebra structure, then $\bigoplus_{p=0}^{N+1} \bigwedge_c^p(V) \otimes \bigwedge_{c^\vee}^p(V^*)$ is a subalgebra.

Proposition 3.3. *The map*

$$\iota = \oplus_{p=0}^{N+1} \iota_p : (\oplus_{p=0}^{N+1} \text{End} \bigwedge_c^p(V), *) \rightarrow \oplus_{p=0}^{N+1} \bigwedge_c^p(V) \otimes \bigwedge_{c^\vee}^p(V^*)$$

is an isomorphism of algebras.

Proof. In $\oplus_{p=0}^{N+1} \bigwedge_c^p(V) \otimes \bigwedge_{c^\vee}^p(V^*)$, we have

$$\begin{cases} (e_i \otimes f_j)(e_i \otimes f_k) &= 0, & \forall i, j, \\ (e_i \otimes f_j)(e_k \otimes f_j) &= 0, & \forall i, j, \\ (e_k \otimes f_j)(e_i \otimes f_l) &= -q^{-1}(e_i \otimes f_j)(e_k \otimes f_l), & \text{if } i < k, \forall j, l, \\ (e_i \otimes f_l)(e_k \otimes f_j) &= -q^{-1}(e_i \otimes f_j)(e_k \otimes f_l), & \text{if } j < l, \forall i, k. \end{cases}$$

It shares the same multiplication rule in (1). And $\oplus_{p=0}^{N+1} \bigwedge_c^p(V) \otimes \bigwedge_{c^\vee}^p(V^*)$ is generated by $e_i \otimes f_j$'s as an algebra. So we get the conclusion. \square

3.3. More information about the q-trace. Using the formula (1), we compute the q-trace on $\oplus_{p=0}^{N+1} \text{End} \bigwedge_c^p(V)$. We also give an inductive formula of the q-trace.

Proposition 3.4. *For any $A \in \text{End} \bigwedge_c^p(V)$ with $A = \sum_{\substack{1 \leq i_1 < \dots < i_p \leq N+1 \\ 1 \leq j_1 < \dots < j_p \leq N+1}} a_{j_1 \dots j_p}^{i_1 \dots i_p} E_{i_1 j_1} * \dots * E_{i_p j_p}$, we have*

$$\begin{aligned} \text{Tr}_q A &= \sum_{w \in \mathfrak{S}_{p, N+1-p}} (-q)^{-2l(w)} a_{w(1) \dots w(p)}^{w(1) \dots w(p)} \\ &= \sum_{1 \leq l_1 < \dots < l_p \leq N+1} q^{(p+1)p-2(l_1+\dots+l_p)} a_{l_1 \dots l_p}^{l_1 \dots l_p}. \end{aligned}$$

In particular, If $A \in \text{End} \bigwedge_c^1(V) = \text{End}(V)$ with $A = \sum a_i^j E_{ji}$, then

$$\text{Tr}_q A = \sum_{i=1}^{N+1} q^{-2(i-1)} a_i^i.$$

Proof. According to the definition, we have

$$\begin{aligned} & A * I_{N+1-p} \\ &= \left(\sum_{\substack{1 \leq i_1 < \dots < i_p \leq N+1 \\ 1 \leq j_1 < \dots < j_p \leq N+1}} a_{j_1 \dots j_p}^{i_1 \dots i_p} E_{i_1 j_1} * \dots * E_{i_p j_p} \right) \\ & \quad * \left(\sum_{1 \leq k_1 < \dots < k_{N+1-p} \leq N+1} E_{k_1 k_1} * \dots * E_{k_{N+1-p} k_{N+1-p}} \right) \\ &= \sum_{\substack{1 \leq i_1 < \dots < i_p \leq N+1 \\ 1 \leq j_1 < \dots < j_p \leq N+1 \\ 1 \leq k_1 < \dots < k_{N+1-p} \leq N+1}} a_{j_1 \dots j_p}^{i_1 \dots i_p} E_{i_1 j_1} * \dots * E_{i_p j_p} * E_{k_1 k_1} * \dots * E_{k_{N+1-p} k_{N+1-p}} \\ &= \sum_{w \in \mathfrak{S}_{p, N+1-p}} a_{w(1) \dots w(p)}^{w(1) \dots w(p)} E_{w(1)w(1)} * \dots * E_{w(N+1)w(N+1)} \\ &= \sum_{w \in \mathfrak{S}_{p, N+1-p}} (-q)^{-2l(w)} a_{w(1) \dots w(p)}^{w(1) \dots w(p)} E_{11} * \dots * E_{N+1, N+1} \end{aligned}$$

$$= \left(\sum_{w \in \mathfrak{S}_{p, N+1-p}} (-q)^{-2l(w)} a_{w(1)\dots w(p)}^{w(1)\dots w(p)} \right) \mathbf{I}_{N+1}.$$

From an easy observation, we know that for any $w \in \mathfrak{S}_{p, N+1-p}$ with $w(1) = l_1, \dots, w(p) = l_p$ we have $l(w) = (l_1-1) + \dots + (l_p-p) = (l_1 + \dots + l_p) - (1+p)p/2$. \square

Proposition 3.5. *For any $A \in \text{End}(V)$ with $Ae_i = \sum_{j=1}^{N+1} a_i^j e_j$, and $0 \leq p \leq N+1$, we have*

$$\text{Tr}_q A^p = \sum_{i=1}^n q^{-2(i-1)} \sum_{j_1, \dots, j_p} a_{j_1}^i a_{j_2}^{j_1} \dots a_{j_p}^{j_{p-1}},$$

and

$$\text{Tr}_q A^{*p} = \sum_{\theta, \tau \in \mathfrak{S}_p} \sum_{w \in \mathfrak{S}_{p, N+1-p}} (-q)^{-(2l(w)+l(\theta)+l(\tau))} a_{\theta w(1)}^{\tau w(1)} \dots a_{\theta w(p)}^{\tau w(p)}.$$

In particular,

$$\text{Tr}_q A^{*N+1} = \sum_{\theta, \tau \in \mathfrak{S}_{N+1}} (-q)^{-l(\theta)-l(\tau)} a_{\theta(1)}^{\tau(1)} \dots a_{\theta(N+1)}^{\tau(N+1)}.$$

Proof. All identities follows from direct computation. \square

For a diagonalizable $A \in \text{End} V$ with $Ae_i = a_i^i e_i$, we have

$$\begin{cases} \text{Tr}_q A^{*N+1} &= (N+1)_{q^{-2}}! a_1^1 \dots a_{N+1}^{N+1}, \\ \text{Tr}_q A^{*p} &= (p)_{q^{-2}}! \sum_{w \in \mathfrak{S}_{p, N+1-p}} (-q)^{-2l(w)} a_{w(1)}^{w(1)} \dots a_{w(p)}^{w(p)}, \\ \text{Tr}_q A^p &= \sum_{i=0}^{N+1} (-q)^{-2(i-1)} (a_i^i)^p. \end{cases}$$

Let $\mathbb{C} = V(1) \subset V(2) \subset \dots \subset V(i) \subset \dots$ be a sequence of vector spaces with $V(i) = \text{Span}_{\mathbb{C}}\{e_1, \dots, e_i\}$. We still use c to denote the action of c restricted on $V(i)$ for all i . For any $1 \leq p \leq N$, we define

$$\begin{aligned} (\text{Tr}_q)_{p+1} : \quad \text{End} \bigwedge_c^{p+1} (V_{N+1}) &\rightarrow \text{End} \bigwedge_c^p (V_N), \\ E_{i_1 j_1} * \dots * E_{i_{p+1} j_{p+1}} &\mapsto E_{i_1 j_1} * \dots * E_{i_p j_p} \text{Tr}_q E_{i_{p+1} j_{p+1}}, \end{aligned}$$

where $1 \leq i_1 < \dots < i_{p+1} \leq N+1$ and $1 \leq j_1 < \dots < j_{p+1} \leq N+1$.

Proposition 3.6. *For any $A \in \text{End} \bigwedge_c^p (V)$, we have*

$$\text{Tr}_q A = (-q)^{p(p-1)} (\text{Tr}_q)_1 (\text{Tr}_q)_2 \dots (\text{Tr}_q)_p A.$$

Proof. We set $A = \sum_{\substack{1 \leq i_1 < \dots < i_p \leq N+1 \\ 1 \leq j_1 < \dots < j_p \leq N+1}} a_{j_1 \dots j_p}^{i_1 \dots i_p} E_{i_1 j_1} * \dots * E_{i_p j_p}$. Then

$$\begin{aligned} & (\text{Tr}_q)_1 (\text{Tr}_q)_2 \dots (\text{Tr}_q)_p A \\ &= \sum_{\substack{1 \leq i_1 < \dots < i_p \leq N+1 \\ 1 \leq j_1 < \dots < j_p \leq N+1}} a_{j_1 \dots j_p}^{i_1 \dots i_p} \text{Tr}_q E_{i_1 j_1} \dots \text{Tr}_q E_{i_p j_p} \\ &= \sum_{\substack{1 \leq i_1 < \dots < i_p \leq N+1 \\ 1 \leq j_1 < \dots < j_p \leq N+1}} a_{j_1 \dots j_p}^{i_1 \dots i_p} \delta_{i_1 j_1} (-q)^{-2(i_1-1)} \dots \delta_{i_p j_p} (-q)^{-2(i_p-1)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{1 \leq i_1 < \dots < i_p \leq N+1} a_{i_1 \dots i_p}^{i_1 \dots i_p} (-q)^{-2(i_1 + \dots + i_p - p)} \\
 &= (-q)^{p(1-p)} \text{Tr}_q A.
 \end{aligned}$$

□

3.4. The relation between q-traces and quantum traces. Now we recall the definition of the quantum trace. For more information, one can see [7].

We know the positive roots of $\mathfrak{sl}_{N+1}(\mathbb{C})$ are

$$\begin{aligned}
 &\alpha_1, \quad \alpha_1 + \alpha_2, \quad \alpha_1 + \alpha_2 + \alpha_3, \quad \dots, \quad \alpha_1 + \dots + \alpha_N, \\
 &\alpha_2, \quad \alpha_2 + \alpha_3, \quad \alpha_2 + \alpha_3 + \alpha_4, \quad \dots, \quad \alpha_2 + \dots + \alpha_N, \\
 &\dots, \\
 &\alpha_N.
 \end{aligned}$$

The sum of all positive roots is

$$\sum_{i=1}^N i(N+1-i)\alpha_i.$$

Set

$$K = K_1^N K_2^{2(N-1)} \dots K_N^N.$$

For any $A \in \text{End}(V)$, we call

$$\text{tr}_q(A) = \text{Tr}(\rho(K)A)$$

the *quantum trace* of A , where Tr is the usual trace of endomorphisms. By direct computation, one gets that if $A \in \text{End}(V)$ with $Ae_i = \sum_{j=1}^{N+1} a_i^j e_j$, then

$$\text{tr}_q(A) = \sum_{i=1}^{N+1} q^{N-2(i-1)} a_i^i.$$

Hence, we get that:

Theorem 3.7. *For any $A \in \text{End}(V)$, we have*

$$\text{Tr}_q A = q^{-N} \text{tr}_q(A).$$

In general, the quantum trace $\text{tr}_q A$ for $A \in \text{End} \bigwedge_c^p(V)$ is defined by:

$$\text{tr}_q A = \text{tr}(\rho^p(K)A),$$

where $\rho^p : \mathcal{U}_q \mathfrak{sl}_{N+1} \rightarrow \text{End} \bigwedge_c^p(V)$ is the representation of $\mathcal{U}_q \mathfrak{sl}_{N+1}$ on $\bigwedge_c^p(V)$ induced by the fundamental representation ρ . For $1 \leq j_1 < \dots < j_p \leq N+1$, we have

$$\begin{aligned}
 &\rho^p(K)A(e_{j_1} \wedge \dots \wedge e_{j_p}) \\
 &= \rho^p(K) \left(\sum_{1 \leq i_1 < \dots < i_p \leq N+1} a_{j_1 \dots j_p}^{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p} \right) \\
 &= \sum_{1 \leq i_1 < \dots < i_p \leq N+1} a_{j_1 \dots j_p}^{i_1 \dots i_p} K e_{i_1} \wedge \dots \wedge K e_{i_p}
 \end{aligned}$$

$$= \sum_{1 \leq i_1 < \dots < i_p \leq N+1} a_{j_1 \dots j_p}^{i_1 \dots i_p} q^{p(N+2)-2(i_1+\dots+i_p)} e_{i_1} \wedge \dots \wedge e_{i_p},$$

where the last equality follows from $K = \text{diag}(q^N, q^{N-2}, \dots, q^{-N})$.

So

$$\text{tr}_q A = \sum_{1 \leq i_1 < \dots < i_p \leq N+1} q^{p(N+2)-2(i_1+\dots+i_p)} a_{i_1 \dots i_p}^{i_1 \dots i_p}.$$

Therefore we have the generalization of the above theorem:

Theorem 3.8. *For any $A \in \text{End } \bigwedge_c^p(V)$, we have*

$$\text{Tr}_q A = q^{-p(N+1-p)} \text{tr}_q A.$$

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